On Classification of Normal Operators in Real Spaces with Indefinite Scalar Product

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Abstract

A real finite dimensional space with indefinite scalar product having v_- negative squares and v_+ positive ones is considered. The paper presents a classification of operators that are normal with respect to this product for the cases $min\{v_-, v_+\} = 1,\ 2$. The approach to be used here was developed in the papers [1] and [2], where the similar classification was obtained for complex spaces with $v = min\{v_-, v_+\} = 1,\ 2$, respectively.

1 Introduction

Consider a real linear space R^n with an indefinite scalar product $[\cdot\,,\cdot]$. By definition, the latter is a nondegenerate sesquilinear Hermitian form. If the ordinary scalar product $(\cdot\,,\cdot)$ is fixed, then there exists a nondegenerate Hermitian operator H such that $[x, y] = (Hx, y)\ \forall x, y \in R^n$. If A is a linear operator ($A : R^n \to R^n$), then the *H-adjoint* of A (denoted by $A^{[*]}$) is defined by the identity $[A^{[*]}x, y] \equiv [x, Ay]$. An operator N is called *H-normal* if $NN^{[*]} = N^{[*]}N$, an operator U is called *H-unitary* if $UU^{[*]} = I$, where I is the identity transformation.

Let V be a nontrivial subspace of R^n. The subspace V is called *neutral* if $[x, y] = 0\ \forall x, y \in V$. If from the conditions $x \in V$ and $\forall y \in V\ [x, y] = 0$ it follows that $x = 0$, then V is called *nondegenerate*. The subspace $V^{[\perp]}$ is defined as the set of all vectors $x \in R^n$: $[x, y] = 0\ \forall y \in V$. If V is nondegenerate, then $V^{[\perp]}$ is also nondegenerate and $V \dot{+} V^{[\perp]} = R^n$.

A linear operator A acting in R^n is called *decomposable* if there exists a nondegenerate subspace $V \subset R^n$ such that both V and $V^{[\perp]}$ are invariant for A or (it is the same) if V is invariant both for A and $A^{[*]}$. Then A is the *orthogonal sum of* $A_1 = A|_V$ *and* $A_2 = A|_{V^{[\perp]}}$. If an operator A is not decomposable, it is called *indecomposable*.

Throughout what follows by a rank of a space we mean $v = \min\{v_-,v_+\}$, where v_- (v_+) is the number of negative (positive) squares of the quadratic form $[x, x]$, i.e., the number of negative (positive) eigenvalues of the operator H. Note that without loss of generality it can be assumed that $v_- \le v_+$ (otherwise H can be replaced by $-H$; the latter (nondegenerate Hermitian operator) has opposite eigenvalues). Later on we assume that $v_- \le v_+$.

The problem is to obtain a complete classification for H-normal operators acting in R^n, i.e., to find a set of canonical forms such that any H-normal operator could be reduced to one and only one of these forms. Since it is sufficient to solve the problem only for indecomposable operators, for any nondegenerate Hermitian matrix H and for any indecomposable H-normal matrix N we would like to point out one and only one of the canonical pairs of matrices $\{\tilde{N}, \tilde{H}\}$ so that the pair $\{N, H\}$ is unitarily similar to $\{\tilde{N}, \tilde{H}\}$ (two pairs of matrices $\{N_1, H_1\}$ and $\{N_2, H_2\}$, where H_1 and H_2 are nondegenerate Hermitian matrices, are called *unitarily similar* if $N_2 = T^{-1}N_1T$, $H_2 = T^*H_1T$ for some invertible matrix T; if $H_1 = H_2$, then they are *H_1-unitarily similar*). In what follows such a classification is presented for operators acting in spaces of rank 1 and 2. As in [2], we will denote by I_r the identity matrix of order $r \times r$, by D_r the $r \times r$ matrix with

1's on the secondary diagonal and zeros elsewhere, and by $A \oplus B \oplus \ldots \oplus C$ a block diagonal matrix with blocks A, B, …, C.

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2 On Decomposition of H-normal Operators in Real Spaces

Let an H-normal operator N act in R^n and have p distinct real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_p$ and q distinct pairs of complex conjugate eigenvalues $\alpha_{p+1} \pm i\beta_{p+1}, \alpha_{p+2} \pm i\beta_{p+2}, \ldots, \alpha_{p+q} \pm i\beta_{p+q}$. Let us define

$$\varphi_k(\lambda) = \left\{ \begin{array}{cl} (\lambda - \lambda_k)^n, & \text{if} \quad 1 \le k \le p \\ (\lambda^2 - 2\alpha_k \lambda + \alpha_k^2 + \beta_k^2)^n, & \text{if} \quad p < k \le p+q, \end{array} \right.$$

$$Q_{ij} = \{x : \ \varphi_i(N)x = \varphi_j(N^{[*]})x = 0\}, \ \ i, j = 1, \ldots, p+q,$$

$$\Omega = \{(i,j) : \ Q_{ij} \ne \{0\}\}.$$

Proposition 1 *The subspaces Q_{ij} have the following properties: (1) $Q_{ij} \cap Q_{kl} = \{0\} \ \ \forall\, (i,j) \ne (k,l)$.*

(2) $\sum_{(i,j)\in\Omega} Q_{ij} = R^n$.

(3) Each subspace Q_{ij} is invariant for both N and $N^{[*]}$. (4) Eigenvalues of the operator $N|_{Q_{ij}}$ are roots of $\varphi_i(\lambda)$, those of the operator $N^{[*]}|_{Q_{ij}}$ are roots of $\varphi_j(\lambda)$. (5) $[Q_{ij}, Q_{kl}] = 0 \ \ \forall\, (i,j) \ne (l,k)$.

Proof:

(1) Suppose $(i,j) \ne (k,l)$. Without loss of generality it can be assumed that $i \ne k$. Let $\exists x : x \in Q_{ij}$, $x \in Q_{kl}$, i.e., $\varphi_i(N)x = \varphi_k(N)x = 0$. Since the polynomials $\varphi_i(\lambda)$ and $\varphi_k(\lambda)$ are relatively prime, there exist polynomials $\psi_i(\lambda)$, $\psi_k(\lambda)$ such that the matrix identity $I \equiv \psi_i(A)\varphi_i(A) + \psi_k(A)\varphi_k(A)$ is valid. Consequently, $x = \psi_i(N)\varphi_i(N)x + \psi_k(N)\varphi_k(N)x = 0$.

(2) The greatest common divisor of the polynomials $\xi_1(\lambda) = \prod_{i\ne 1} \varphi_i(\lambda)$, $\xi_2(\lambda) = \prod_{i \ne 2} \varphi_i(\lambda)$, …, $\xi_{p+q}(\lambda) = \prod_{i \ne p+q} \varphi_i(\lambda)$ is equal to 1, therefore, there exist polinomials $\psi_1(\lambda)$, $\psi_2(\lambda)$, …, $\psi_{p+q}(\lambda)$ such that $I = \sum_{i=1}^{p+q} \psi_i(A)\xi_i(A) \ \ \forall A$. Hence, $\forall x \ x = \sum_{i=1}^{p+q} \psi_i(N)\xi_i(N)x = \sum_{i=1}^{p+q} x_i$ (where $x_i = \psi_i(N)\xi_i(N)x$). Since the product of all $\varphi_i(\lambda)$ annihilates N, we have $\varphi_i(N)x_i = 0 \ \forall i$, i.e., $R^n = \sum_{i=1}^{p+q} Q_i$, where $Q_i = \{x : \varphi_i(x) = 0\}$. Similarly, each subspace Q_i is a direct sum of the subspaces $Q_{ij} = \{x \in Q_i : \ \varphi_j(N^{[*]})x = 0\}$. Disregarding the trivial subspaces Q_{ij}, we obtain the desired equality $R^n = \sum_{(i,j)\in\Omega} Q_{ij}$.

(3) Since N and $N^{[*]}$ commute, for all (i,j) and $x \in Q_{ij}$ we have $0 = N\varphi_i(N)x = \varphi_i(N)Nx$, $0 = N\varphi_j(N^{[*]})x = \varphi_j(N^{[*]})Nx$, i.e., $Nx \in Q_{ij}$. It can be checked in the same way that $N^{[*]}x \in Q_{ij}$.

(4) Let $N|_{Q_{ij}}$ have an eigenvalue λ_0 such that $\varphi_i(\lambda_0) \ne 0$. Then there exists a (real or complex) eigenvector $x \ne 0$ corresponding to the eigenvalue λ_0. Since the polynomials $\lambda - \lambda_0$ and $\varphi_i(\lambda)$ are relatively prime, there exist polynomials $\psi_1(\lambda)$, $\psi_2(\lambda)$ such that the identity $I = \psi_1(A)(A - \lambda_0 I) + \psi_2(A)\varphi_i(A)$ holds for all (complex) matrices A. Consequently, $x = \psi_1(N)(N - \lambda_0 I)x + \psi_2(N)\varphi_i(N)x = 0$ because $(N - \lambda_0 I)x = \varphi_i(N)x = 0$. The contradiction obtained shows that all eigenvalues of $N|_{Q_{ij}}$ are roots of $\varphi_i(\lambda)$. The operator $N^{[*]}|_{Q_{ij}}$ can be considered in the same way.

(5) Let $i \ne l$. Take arbitrary vectors $x \in Q_{ij}$, $y \in Q_{kl}$. Since the eigenvalues of $N|_{Q_{ij}}$ are not roots of $\varphi_l(\lambda)$, the operator $\varphi_l(N)|_{Q_{ij}}$ is nondegenerate. Therefore, $\exists z \in Q_{ij} : \varphi_l(N)z = x$. We have $[x,y] = [\varphi_l(N)z, y] = [z, \varphi_l(N^{[*]})y] = [z, 0] = 0$.

The proof of the proposition is completed.

Now let $V_i = Q_{ii}$ $((i,i) \in \Omega)$, $V_{jk} = span\{Q_{jk}, Q_{kj}\}$ $((j,k) \in \Omega, \ j < k)$. The subspaces V_i, V_{jk} are mutually orthogonal, the intersection of any two of them is zero, and their sum is R^n. It follows from the nondegeneracy of H that each subspace V_i, V_{jk} is nondegenerate. The restriction $N|_{V_i}$ has the only real eigenvalue λ_i if $i \le p$ or the pair of complex conjugate eigenvalues $\alpha_i \pm i\beta_i$ if $i > p$. The restriction $N|_{V_{jk}}$ has two distinct real eigenvalues λ_j, λ_k if $j, k \le p$, one real eigenvalue λ_j and the pair of complex conjugate eigenvalues $\alpha_k \pm i\beta_k$ if $j \le p$, $k > p$, or two distinct pairs $\alpha_j \pm i\beta_j$, $\alpha_k \pm i\beta_k$ if $j, k > p$.

Thus, we have proved the following lemma:

Lemma 1 *Any H-normal operator N acting in R^n is an orthogonal sum of H-normal operators each of which has one of the following sets of eigenvalues:*

(a) one real eigenvalue;

(b) two distinct real eigenvalues;

(c) two complex conjugate eigenvalues;

(d) one real and two complex conjugate eigenvalues;

(e) two distinct pairs of complex conjugate eigenvalues.

This lemma shows the principal difference between real and complex spaces because indecomposable operators acting in complex spaces have either one or two distinct eigenvalues (Lemma 1 from [1]).

3 Classification of H-normal Operators Acting in Spaces of Rank 1

This section is closely related to [1].

Let us classify indecomposable H-normal operators acting in a space R^n of rank 1. According to Lemma 1, we can consider only operators having one of the sets of eigenvalues (a) - (e). However, for a space of rank 1 not all variants are possible, namely, the alternatives (d) and (e) cannot be realized. Indeed, if $N|_{Q_{12}}$ (or $N^{[*]}|_{Q_{12}}$) has two eigenvalues $\alpha \pm i\beta$, the subspace Q_{12} is necessarily of dimension 2 or higher. However, since Q_{12} is neutral, $dim\, Q_{12} \leq 1$. Thus, the alternatives (d) and (e) are impossible. Let us consider the remaining variants and prove the following theorem:

Theorem 1 *If an indecomposable H-normal operator N ($N : R^n \to R^n$) acts in a space with indefinite scalar product having $v_- = 1$ negative squares and $v_+ \geq 1$ positive ones, then $2 \leq n \leq 4$ and the pair $\{N, H\}$ is unitarily similar to one and only one of canonical pairs (1), (2), (3), (4), (5), (6):*

$$N = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \;\; \lambda_1 < \lambda_2, \;\; H = D_2, \tag{1}$$

$$N = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \;\; \beta > 0, \;\; H = D_2, \tag{2}$$

$$N = \begin{pmatrix} \lambda & z \\ 0 & \lambda \end{pmatrix}, \;\; z = \pm 1, \;\; H = D_2, \tag{3}$$

$$N = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}, \;\; H = D_3, \tag{4}$$

$$N = \begin{pmatrix} \lambda & 1 & r \\ 0 & \lambda & -1 \\ 0 & 0 & \lambda \end{pmatrix}, \;\; H = D_3, \tag{5}$$

$$N = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & \cos\alpha \\ 0 & 0 & \lambda & \sin\alpha \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \; 0 < \alpha < \pi, \;\; H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_2 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{6}$$

The proof of the theorem is presented in the following subsections.

3.1 One Real Eigenvalue of N

Let us take advantage of Proposition 1 from [2], which is proved for complex spaces but is valid for real ones as well: *If an indecomposable H-normal operator $N: R^n \to R^n$ ($n > 1$) has the only eigenvalue λ, then there exists a decomposition of R^n into a direct sum of subspaces*

$$S_0 = \{x : (N - \lambda I)x = (N^{[*]} - \lambda I)x = 0\}, \tag{7}$$

S, S_1 such that

$$N = \begin{pmatrix} N' = \lambda I & * & * \\ 0 & N_1 & * \\ 0 & 0 & N'' = \lambda I \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 & I \\ 0 & H_1 & 0 \\ I & 0 & 0 \end{pmatrix}, \tag{8}$$

where $N' : S_0 \to S_0$, $N_1 : S \to S$, $N'' : S_1 \to S_1$, the internal operator N_1 is H_1-normal, and the pair $\{N_1, H_1\}$ is determined up to unitary similarity. To go over from one decomposition $R^n = S_0 \dot{+} S \dot{+} S_1$ to another by a transformation T it is necessary that the matrix T be block triangular with respect to both decompositions.

Since S_0 is neutral, $dim S_0 = 1$. According to Proposition 2 from [2], if the subspace S_0 is one-dimensional, then the operator N is indecomposable. So, it is not necessary to check the indecomposability for each canonical form to be obtained in this subsection. As H has one negative eigenvalue, H_1 has only positive eigenvalues and one can assume that $H_1 = I$, $N_1 = \lambda I$. Later on we will no longer stipulate that $H_1 = I$, $N_1 = \lambda I$. By Theorem 1 of [2] (it is also valid for real spaces), $n \leq 4$. Consider the cases $n = 2, 3, 4$ successively.

3.1.1 $n = 2$

The matrices N and H have form (8):

$$N = \begin{pmatrix} \lambda & a \\ 0 & \lambda \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since $S_0 \cap S_1 = \{0\}$, $a \neq 0$. Let $\tilde{v_1} = \sqrt{|a|} v_1$, $\tilde{v_2} = 1/\sqrt{|a|} v_2$. Then we do not change the matrix H and reduce N to form (3). Since (3) is a special case of canonical form (16) from Theorem 1 ([1]), the number z is an H-unitary invariant, i.e., two forms (3) with different values of z are not H-unitarily similar.

3.1.2 $n = 3$

The matrices N and H have form (8):

$$N = \begin{pmatrix} \lambda & a & b \\ 0 & \lambda & c \\ 0 & 0 & \lambda \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The condition of the H-normality of N is

$$a^2 = c^2.$$

If $a = 0$, then $c = 0$ and $v_2 \in S_0$, which is impossible because of the condition $S_0 \cap S = \{0\}$. Therefore, $a \neq 0$. Let $\tilde{v_1} = a v_1$, $\tilde{v_3} = 1/a\, v_3$. then we reduce N to the form

$$N = \begin{pmatrix} \lambda & 1 & b' \\ 0 & \lambda & x \\ 0 & 0 & \lambda \end{pmatrix}, \quad x = \pm 1$$

without changing the matrix H. If $x = 1$, take the H-unitary transformation T (throughout what follows only H-unitary transformations are used unless otherwise stipulated):

$$T = \begin{pmatrix} 1 & \frac{1}{2}b' & -\frac{1}{8}b'^2 \\ 0 & 1 & -\frac{1}{2}b' \\ 0 & 0 & 1 \end{pmatrix}.$$

It reduces N to form (4). If $x = -1$, the number b' turns out to be H-unitary invariant. Indeed, let

$$N - \lambda I = \begin{pmatrix} 0 & 1 & r \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{N} - \lambda I = \begin{pmatrix} 0 & 1 & \tilde{r} \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$

and some matrix $T = \{t_{ij}\}_{i,j=1}^{3}$ satisfy The conditions

$$\begin{aligned} NT &= T\tilde{N}, && (9) \\ tT^{[*]} &= I. && (10) \end{aligned}$$

Then, according to Proposition 1 from [2], T is block triangular with respect to the decomposition $R^n = S_0 \dot{+} S \dot{+} S_1$, i.e., upper triangular. Condition (9) implies

$$\begin{gathered} t_{11} = t_{22} = t_{33}, \\ t_{23} + rt_{33} = \tilde{r}t_{11} - t_{12}. \end{gathered} \tag{11}$$

Since the diagonal terms of T are equal to each other, From (10) it follows that $t_{12} + t_{23} = 0$. Then from (11) we get $r = \tilde{r}$, Q.E.D. The forms obtained are not H-unitarily similar. Indeed, let an H-unitary matrix $T = \{t_{ij}\}_{i,j=1}^{3}$ reduce the first form to the second. Since T is upper triangular (Proposition 1 from [2]), from (9) it follows that $t_{11} = t_{22} = -t_{33}$, which is impossible because condition (10) implies $t_{11}t_{33} = 1$. Thus, we have obtained two canonical forms: (4) and (5).

3.1.3 $n = 4$

The matrices N and H have form (8):

$$N = \begin{pmatrix} \lambda & a & b & c \\ 0 & \lambda & 0 & d \\ 0 & 0 & \lambda & e \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The condition of the H-normality of N is

$$a^2 + b^2 = d^2 + e^2. \tag{12}$$

Since $a^2 + b^2 \neq 0$ (otherwise $v_2, v_3 \in S_0$, which is impossible), without loss of generality it can be assumed that $a \neq 0$. Taking $\widetilde{v_1} = av_1$, $\widetilde{v_4} = v_4/a$, we reduce N to the form

$$N = \begin{pmatrix} \lambda & 1 & b' & c' \\ 0 & \lambda & 0 & d' \\ 0 & 0 & \lambda & e' \\ 0 & 0 & 0 & \lambda \end{pmatrix}.$$

Further, let us apply the transformation

$$T = \begin{pmatrix} \sqrt{1+b'^2} & 0 & 0 & 0 \\ 0 & 1/\sqrt{1+b'^2} & -b'/\sqrt{1+b'^2} & 0 \\ 0 & b'/\sqrt{1+b'^2} & 1/\sqrt{1+b'^2} & 0 \\ 0 & 0 & 0 & 1/\sqrt{1+b'^2} \end{pmatrix}.$$

Then we get

$$N = \begin{pmatrix} \lambda & 1 & 0 & c'' \\ 0 & \lambda & 0 & d'' \\ 0 & 0 & \lambda & e'' \\ 0 & 0 & 0 & \lambda \end{pmatrix}.$$

Note that $e'' \neq 0$ because otherwise $v_3 \in S_0$, which is impossible because $S_0 \cap S = \{0\}$. The number e'' can be replaced by $-e''$ by means of the (H-unitary) transformation $\widetilde{v_3} = -v_3$. So, we can assume $e'' > 0$. Moreover, it can be assumed that $c'' = 0$. To this end it is sufficient to take the transformation

$$T = \begin{pmatrix} 1 & 0 & c''/e'' & -\frac{1}{2}c''^2/e''^2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -c''/e'' \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then c'' will vanish, d'' and e'' will not change. Condition (12) of the H-normality of N implies $d'' = \cos\alpha$, $e'' = \sin\alpha$ ($\alpha \in (0;\pi)$). Show the H-unitary invariance of the parameter α. Let an H-unitary matrix $T = \{t_{ij}\}_{i,j=1}^4$ reduce N to the form

$$\tilde{N} = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & \cos\tilde{\alpha} \\ 0 & 0 & \lambda & \sin\tilde{\alpha} \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \ \tilde{\alpha} \in (0;\pi).$$

Then, according to Proposition 1 from [2], T is block triangular with respect to the decomposition $R^n = S_0 \dot{+} S \dot{+} S_1$ and from (9) it follows that $t_{23} = 0$. Now condition (10) yields $t_{32} = 0$. Applying (9) again, we have

$$\begin{aligned} t_{11} &= t_{22}, \\ t_{44}\cos\alpha &= t_{22}\cos\tilde{\alpha}, \\ t_{44}\sin\alpha &= t_{33}\sin\tilde{\alpha}. \end{aligned}$$

Condition (10) yields $t_{11}t_{44} = t_{22}^2 = t_{33}^2 = 1$ so that $t_{11} = t_{22} = t_{44} = \pm 1$. Hence, $\cos\alpha = \cos\tilde{\alpha}$. Since $\sin\alpha,\ \sin\tilde{\alpha} > 0$, we have $t_{33} = t_{44}$ and $\sin\alpha = \sin\tilde{\alpha}$. Consequently, $\tilde{\alpha} = \alpha$, Q.E.D. Thus, we have obtained canonical form (6).

3.2 Two Distinct Real Eigenvalues of N

According to Proposition 1, in this case

$$N = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}, \ a \neq 0.$$

It can be assumed that $a = 1$ (to this end it is sufficient to take $\widetilde{v_1} = v_1/a$, $\widetilde{v_2} = v_2$). Since the order of eigenvalues is not fixed, we can assume that $\lambda_1 < \lambda_2$. Thus, we have obtained canonical pair (1).

3.3 Two Complex Conjugate Eigenvalues of N

Let N have two distinct eigenvalues $\lambda = \alpha + i\beta$, $\overline{\lambda} = \alpha - i\beta$. Since N and $N^{[*]}$ commute, there exists a vector $z = x + iy$ ($x, y \in R^n$) such that either $Nz = \lambda z$, $N^{[*]}z = \overline{\lambda}z$ or $Nz = \lambda z$, $N^{[*]}z = \lambda z$. In the first case $[z,\overline{z}] = 0$. Indeed, $\overline{\lambda}[z,\overline{z}] = [\lambda z,\overline{z}] = [Nz,\overline{z}] = [z, N^{[*]}\overline{z}] = [z,\lambda\overline{z}] = \lambda[z,\overline{z}]$. Therefore, $(\lambda - \overline{\lambda})[z,\overline{z}] = 0$, hence $[z,\overline{z}] = 0$. Let us write in detail the condition obtained: $[x+iy, x-iy] = [x,x] - i[y,x] - i[x,y] - [y,y] = 0$, i.e., $[x,y] = 0$, $[x,x] = [y,y]$. Since two-dimensional subspace $V = span\{x,y\}$ cannot be neutral, we have $[x,x] \neq 0$. Thus, V is a nondegenerate subspace which is invariant for N and $N^{[*]}$. For N to be

indecomposable it is necessary to have $R^n = V$. But $[x, x] = [y, y]$, i.e., H is either positive or negative definite, which contradicts the condition $min\{v_-, v_+\} = 1$. Thus, only the case $Nz = \lambda z$, $N^{[*]}z = \lambda z$ is possible. It can be shown as before that $[z, z] = 0$, i.e., $[x, x] = -[y, y]$ so that the subspace $V = span\{x, y\}$ is either nondegenerate or neutral. As above, we see that V is necessarily nondegenerate and $V = R^n$.

Thus, for the basis $\{x, y\}$ we have

$$N = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \quad H = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \ (a^2 + b^2 \neq 0).$$

Let us reduce H to the form D_2 without changing the matrix N. To this end it is sufficient to take

$$T = \begin{pmatrix} t_{11} & t_{12} \\ -t_{12} & t_{11} \end{pmatrix},$$

where

$$\begin{aligned} -2t_{11}t_{12} &= a, \\ t_{11}^2 - t_{12}^2 &= b \end{aligned}$$

(it can be checked that this system always has a real solution $\{t_{11}, t_{12}\}$). Then

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix} = T^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} T, \quad TN = NT.$$

One can replace β by $-\beta$ by means of the H-unitary transformation $T = D_2$, therefore, one can assume that $\beta > 0$. Thus, we have obtained canonical pair (2). The proof of Theorem 1 is completed.

4 Classification of H-normal Operators Acting in Spaces of Rank 2

The objective of this section is to prove the following theorem (the subspace S_0 and the internal operator N_1 are defined in Section 3.1 by formulas (7), (8), respectively):

Theorem 2 *If an indecomposable H-normal operator N ($N : R^n \to R^n$) acts in a space with indefinite scalar product having $v_- = 2$ negative squares and $v_+ \geq 2$ positive ones, then $4 \leq n \leq 8$ and the pair $\{N, H\}$ is unitarily similar to one and only one of the canonical pairs $\{$(13),(14)$\}$ - $\{$(54),(55)$\}$. The list of all the canonical pairs is as follows.*

If N has one real eigenvalue λ, $\dim S_0 = 1$, the internal operator N_1 is indecomposable, and $n = 4$, then the pair $\{N, H\}$ is unitarily similar to the canonical pair $\{$(13),(14)$\}$:

$$N = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & z & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad z = \pm 1, \tag{13}$$

$$H = D_4. \tag{14}$$

If N has one real eigenvalue λ, $\dim S_0 = 1$, N_1 is indecomposable, and $n = 5$, then the pair $\{N, H\}$ is unitarily similar to one and only one of the canonical pairs $\{$(15),(17)$\}$, $\{$(16),(17)$\}$:

$$N = \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \tag{15}$$

$$N = \begin{pmatrix} \lambda & 1 & -r_1 & 0 & r_2 \\ 0 & \lambda & 1 & r_1 & 0 \\ 0 & 0 & \lambda & -1 & -r_1 \\ 0 & 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \tag{16}$$

$$H = D_5. \tag{17}$$

If N has one real eigenvalue λ, $\dim S_0 = 1$, N_1 is decomposable, and $n = 4$, then the pair $\{N, H\}$ is unitarily similar to one and only one of the canonical pairs $\{(18),(20)\}$, $\{(19),(20)\}$:

$$N = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & z \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \; z = \pm 1, \tag{18}$$

$$N = \begin{pmatrix} \lambda & 1 & z & 0 \\ 0 & \lambda & 0 & r \\ 0 & 0 & \lambda & z/r \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \; z = \pm 1, \; |r| > 1, \tag{19}$$

$$H = D_4. \tag{20}$$

If N has one real eigenvalue λ, $\dim S_0 = 1$, N_1 is decomposable, and $n = 5$, then the pair $\{N, H\}$ is unitarily similar to the canonical pair $\{(21),(22)\}$:

$$N = \begin{pmatrix} \lambda & 1 & 0 & \frac{1}{2}r^2 & 0 \\ 0 & \lambda & 0 & z & 0 \\ 0 & 0 & \lambda & 0 & r \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \; z = \pm 1, \; r > 0, \tag{21}$$

$$H = D_5. \tag{22}$$

If N has one real eigenvalue λ, $\dim S_0 = 1$, N_1 is decomposable, and $n = 6$, then the pair $\{N, H\}$ is unitarily similar to one and only one of the canonical pairs $\{(23),(25)\}$, $\{(24),(25)\}$:

$$N = \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 & -r^2/2 \\ 0 & 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 1 \\ 0 & 0 & 0 & 0 & \lambda & r \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \; r > 0, \tag{23}$$

$$N = \begin{pmatrix} \lambda & 1 & -2r_1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & r_1 & 0 & -2r_1^2 + r_2^2/2 \\ 0 & 0 & \lambda & -1 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & -1 \\ 0 & 0 & 0 & 0 & \lambda & r_2 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \; r_2 > 0, \tag{24}$$

$$H = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & D_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{25}$$

If N has one real eigenvalue λ, $\dim S_0 = 2$, and $n = 4$, then the pair $\{N, H\}$ is unitarily similar to one and only one of the canonical pairs $\{$(26),(30)$\}$, $\{$(27),(30)$\}$, $\{$(28),(30)$\}$, $\{$(29),(30)$\}$:

$$N = \begin{pmatrix} \lambda & 0 & \cos\alpha & \sin\alpha \\ 0 & \lambda & -\sin\alpha & \cos\alpha \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix},\ 0 < \alpha < \pi, \tag{26}$$

$$N = \begin{pmatrix} \lambda & 0 & 0 & 1 \\ 0 & \lambda & r & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix},\ |r| > 1, \tag{27}$$

$$N = \begin{pmatrix} \lambda & 0 & \frac{1}{2}z & z \\ 0 & \lambda & -z & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix},\ z = \pm 1, \tag{28}$$

$$N = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \tag{29}$$

$$H = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}. \tag{30}$$

If N has one real eigenvalue λ, $\dim S_0 = 2$, and $n = 5$, then the pair $\{N, H\}$ is unitarily similar to one and only one of the canonical pairs $\{$(31),(33)$\}$, $\{$(32),(33)$\}$:

$$N = \begin{pmatrix} \lambda & 0 & 1 & 0 & 0 \\ 0 & \lambda & 0 & 1 & 0 \\ 0 & 0 & \lambda & z & 0 \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix},\ z = \pm 1, \tag{31}$$

$$N = \begin{pmatrix} \lambda & 0 & 1 & 0 & 0 \\ 0 & \lambda & 0 & r & z \\ 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix},\ z = \pm 1,\ r > 0, \tag{32}$$

$$H = \begin{pmatrix} 0 & 0 & I_2 \\ 0 & I_1 & 0 \\ I_2 & 0 & 0 \end{pmatrix}. \tag{33}$$

If N has one real eigenvalue λ, $\dim S_0 = 2$, and $n = 6$, then the pair $\{N, H\}$ is unitarily similar to one and only one of the canonical pairs $\{$(34),(36)$\}$, $\{$(35),(36)$\}$:

$$N = \begin{pmatrix} \lambda & 0 & 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 1 & r & 0 \\ 0 & 0 & \lambda & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 1 \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix},\ r > 0, \tag{34}$$

$$N=\begin{pmatrix} \lambda & 0 & 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 1 & r & 0 \\ 0 & 0 & \lambda & 0 & \cos\alpha & \sin\alpha \\ 0 & 0 & 0 & \lambda & -\sin\alpha & \cos\alpha \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \ \ 0<\alpha<\pi, \tag{35}$$

$$H=\begin{pmatrix} 0 & 0 & I_2 \\ 0 & I_2 & 0 \\ I_2 & 0 & 0 \end{pmatrix}. \tag{36}$$

If N has one real eigenvalue λ, dim $S_0=2$, and $n=7$, then the pair $\{N,H\}$ is unitarily similar to the canonical pair $\{$(37),(38)$\}$:

$$N=\begin{pmatrix} \lambda & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & \cos\alpha & -\sin\alpha\cos\beta \\ 0 & 0 & 0 & \lambda & 0 & \sin\alpha & \cos\alpha\cos\beta \\ 0 & 0 & 0 & 0 & \lambda & 0 & \sin\beta \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}, \ 0<\alpha,\beta<\pi, \tag{37}$$

$$H=\begin{pmatrix} 0 & 0 & I_2 \\ 0 & I_3 & 0 \\ I_2 & 0 & 0 \end{pmatrix}. \tag{38}$$

If N has one real eigenvalue λ, dim $S_0=2$, and $n=8$, then the pair $\{N,H\}$ is unitarily similar to one and only one of the canonical pairs $\{$(39),(41)$\}$, $\{$(40),(41)$\}$:

$$N=\begin{pmatrix} \lambda & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & \cos\alpha\sin\beta & \sin\alpha\sin\beta \\ 0 & 0 & 0 & \lambda & 0 & 0 & -\sin\alpha\sin\beta & \cos\alpha\sin\beta \\ 0 & 0 & 0 & 0 & \lambda & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & \cos\beta \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix},$$

$$0<\alpha<\pi, \ \ 0<\beta<\pi/2, \tag{39}$$

$$N=\begin{pmatrix} \lambda & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & \cos\alpha\sin\beta & \sin\alpha\sin\gamma \\ 0 & 0 & 0 & \lambda & 0 & 0 & -\sin\alpha\sin\beta & \cos\alpha\sin\gamma \\ 0 & 0 & 0 & 0 & \lambda & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & \cos\gamma \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix},$$

$$0<\alpha<\pi, \ \ 0\le\gamma<\beta<\pi/2, \tag{40}$$

$$H=\begin{pmatrix} 0 & 0 & I_2 \\ 0 & I_4 & 0 \\ I_2 & 0 & 0 \end{pmatrix}. \tag{41}$$

If N has 2 distinct real eigenvalues λ_1, λ_2, then the pair $\{N, H\}$ is unitarily similar to the canonical pair $\{(42),(43)\}$:

$$N = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & r & \lambda_2 \end{pmatrix}, \quad for\ r \neq 0\ \ \lambda_1 < \lambda_2, \tag{42}$$

$$H = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}. \tag{43}$$

If N has 3 eigenvalues: $\lambda \in R$, $\alpha \pm i\beta$ ($\alpha, \beta \in \Re$, $\beta > 0$), then the pair $\{N, H\}$ is unitarily similar to the canonical pair $\{(44),(45)\}$:

$$N = \begin{pmatrix} \alpha & \beta & 0 & 0 \\ -\beta & \alpha & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \tag{44}$$

$$H = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}. \tag{45}$$

If N has 4 eigenvalues: $\alpha_1 \pm i\beta_1$, $\alpha_2 \pm i\beta_2$, ($\alpha_1, \beta_1, \alpha_2, \beta_2 \in \Re$, $0 < \beta_1 \leq \beta_2$, $\alpha_1 < \alpha_2$ if $\beta_1 = \beta_2$), then the pair $\{N, H\}$ is unitarily similar to the canonical pair $\{(46),(47)\}$:

$$N = \begin{pmatrix} \alpha_1 & \beta_1 & 0 & 0 \\ -\beta_1 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_2 & z\beta_2 \\ 0 & 0 & -z\beta_2 & \alpha_2 \end{pmatrix}, \ z = \pm 1, \tag{46}$$

$$H = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}. \tag{47}$$

If N has 2 eigenvalues $\alpha \pm i\beta$ ($\alpha, \beta \in \Re$, $\beta > 0$), and $n = 4$, then the pair $\{N, H\}$ is unitarily similar to one and only one of the canonical pairs $\{(48),(50)\}$, $\{(49),(50)\}$:

$$N = \begin{pmatrix} \alpha & \beta & \cos\gamma & \sin\gamma \\ -\beta & \alpha & -\sin\gamma & \cos\gamma \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta & \alpha \end{pmatrix}, \ 0 \leq \gamma < 2\pi, \tag{48}$$

$$N = \begin{pmatrix} \alpha & \beta & 0 & 1 \\ -\beta & \alpha & 1 & 0 \\ 0 & 0 & \alpha & -\beta \\ 0 & 0 & \beta & \alpha \end{pmatrix}, \tag{49}$$

$$H = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}. \tag{50}$$

If N has 2 eigenvalues $\alpha \pm i\beta$ ($\alpha, \beta \in \Re$, $\beta > 0$), and $n = 6$, then the pair $\{N, H\}$ is unitarily similar to one and only one of the canonical pairs $\{(51),(53)\}$, $\{(52),(53)\}$:

$$N = \begin{pmatrix} \alpha & \beta & 0 & 0 & 0 & r \\ -\beta & \alpha & 0 & 1 & (\cos\gamma + 1)/4\beta - r & \sin\gamma/4\beta \\ 0 & 0 & \alpha & \beta & \frac{1}{2}(\cos\gamma + 1) & \frac{1}{2}\sin\gamma \\ 0 & 0 & -\beta & \alpha & -\frac{1}{2}\sin\gamma & \frac{1}{2}(\cos\gamma - 1) \\ 0 & 0 & 0 & 0 & \alpha & \beta \\ 0 & 0 & 0 & 0 & -\beta & \alpha \end{pmatrix},$$

$$0 \leq \gamma < 2\pi,\ \gamma \neq \pi, \tag{51}$$

$$N = \left(\begin{array}{cccccc} \alpha & \beta & 0 & 0 & r & 0 \\ -\beta & \alpha & 0 & 1 & 0 & r \\ 0 & 0 & \alpha & \beta & 0 & 0 \\ 0 & 0 & -\beta & \alpha & 0 & -1 \\ 0 & 0 & 0 & 0 & \alpha & \beta \\ 0 & 0 & 0 & 0 & -\beta & \alpha \end{array} \right), \tag{52}$$

$$H = \left(\begin{array}{ccc} 0 & 0 & I_2 \\ 0 & I_2 & 0 \\ I_2 & 0 & 0 \end{array} \right). \tag{53}$$

If N has 2 eigenvalues $\alpha \pm i\beta$ ($\alpha, \beta \in \Re$, $\beta > 0$), and $n = 8$, then the pair $\{N, H\}$ is unitarily similar to the canonical pair $\{(54),(55)\}$:

$$N = \left(\begin{array}{cccccccc} \alpha & \beta & 0 & 0 & 0 & 0 & 0 & 0 \\ -\beta & \alpha & 0 & 1 & 0 & 0 & \sin^2\gamma/2\beta & \sin\gamma\cos\gamma\cos\delta/2\beta \\ 0 & 0 & \alpha & \beta & 0 & 0 & \sin^2\gamma & \sin\gamma\cos\gamma\cos\delta \\ 0 & 0 & -\beta & \alpha & 0 & 0 & -\sin\gamma\cos\gamma\cos\delta & -\cos^2\gamma \\ 0 & 0 & 0 & 0 & \alpha & \beta & \sin\gamma\cos\gamma\sin\delta & 0 \\ 0 & 0 & 0 & 0 & -\beta & \alpha & 0 & \sin\gamma\cos\gamma\sin\delta \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha & \beta \\ 0 & 0 & 0 & 0 & 0 & 0 & -\beta & \alpha \end{array} \right)$$

$$0 < \gamma < \pi/2,\ 0 < \delta < \pi, \tag{54}$$

$$H = \left(\begin{array}{ccc} 0 & 0 & I_2 \\ 0 & I_4 & 0 \\ I_2 & 0 & 0 \end{array} \right). \tag{55}$$

Here all parameters are H-unitary invariants, i.e., the same canonical forms are H-unitarily similar to each other iff the values of all parameters are equal.

The proof of the theorem is presented in what follows.

4.1 One Real Eigenvalue of N

The case when N has only one real eigenvalue λ can be considered as in [2]. Namely, if $dim\ S_0 = 1$, then there exists two alternatives: N_1 is indecomposable or decomposable, this property being independent of the choice of the decomposition $R^n = S_0 \dot{+} S \dot{+} S_1$ because the indecomposability or decomposability of N_1 does not change under unitary similarity of the pair $\{N_1, H_1\}$. In the former case one can show that $n \leq 5$ and obtain the canonical forms $\{(13),(14)\}$ - $\{(16),(17)\}$, in the latter one can show that $n \leq 6$ and obtain the canonical forms $\{(18),(20)\}$ - $\{(24),(25)\}$ in just the same way as it was done in [2]. If the subspace S_0 is two-dimensional, the operator N can also be considered as in [2] except for the case $n = 4$ because one of the corresponding canonical forms in [2] is essentially complex. Thus, for the case when N has one real eigenvalue λ we will consider only the alternative $dim\ S_0 = 2$, $n = 4$ and omit the rest.

4.1.1 $dim\ S_0 = 2$, $n = 4$

In this case $R^4 = S_0 \dot{+} S_1$. Therefore,

$$N - \lambda I = \left(\begin{array}{cc} 0 & N_2 \\ 0 & 0 \end{array} \right) = \left(\begin{array}{cccc} 0 & 0 & a & b \\ 0 & 0 & c & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right),\quad H = \left(\begin{array}{cc} 0 & I_2 \\ I_2 & 0 \end{array} \right),$$

and the submatrix N_2 is not restricted by the condition of the H-normality of N.

(a) $detN_2 \neq 0$. Suppose an H-unitary transformation

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$$

reduces $N - \lambda I$ to the form $\tilde{N} - \lambda I$:

$$N - \lambda I = \begin{pmatrix} 0 & N_2 \\ 0 & 0 \end{pmatrix}, \quad \tilde{N} - \lambda I = \begin{pmatrix} 0 & \widetilde{N_2} \\ 0 & 0 \end{pmatrix}.$$

Then conditions (56) - (58) below are necessarily satisfied:

$$\begin{aligned} N_2 T_3 &= 0, & (56) \\ N_2 T_4 &= T_1 \widetilde{N_2}, & (57) \\ 0 &= T_3 \widetilde{N_2}. & (58) \end{aligned}$$

Since N_2 is nondegenerate, (56) is satisfied only if $T_3 = 0$. The operator T is H-unitary iff

$$\begin{aligned} T_1 T_4^* &= I, & (59) \\ T_1 T_2^* + T_2 T_1^* &= 0. & (60) \end{aligned}$$

It follows from system (59) - (60) that without loss of generality we can consider only quasidiagonal transformations $T = T_1 \oplus T_1^{*-1}$ because T_2 does not appear in equations (56) - (58).

Thus, the only condition

$$N_2 = T_1 \widetilde{N_2} T_1^* \tag{61}$$

should be satisfied, i.e., it is necessary to find out what form a nondegenerate 2×2-matrix N_2 can be reduced to under congruence.

Consider the matrix $N_2' = N_2 N_2^{*-1}$. Its spectral characteristics are invariant because $N_2' = T_1 \widetilde{N_2}' T_1^{-1}$. Since $det\, N_2' = 1$, N_2' has either two complex conjugate eigenvalues $\cos\alpha \pm i \sin\alpha$ or two real eigenvalues r, $1/r$ ($r \neq 0$). In the former case N_2' can be reduced to the form

$$N_2' = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}, \quad 0 < \alpha < \pi, \tag{62}$$

in the latter to the Jordan normal form.

If N_2' has form (62), then

$$N_2 = \begin{pmatrix} t\, \sin\alpha/(1-\cos\alpha) & t \\ -t & t\, \sin\alpha/(1-\cos\alpha) \end{pmatrix}, \ t \neq 0.$$

As $det\, N_2 = 2t^2/(1-\cos\alpha) > 0$, one can take $T_1 = \sqrt{det\, N_2} I$ and obtain

$$N_2 = \begin{pmatrix} \pm\cos\frac{\alpha}{2} & \pm\sin\frac{\alpha}{2} \\ \mp\sin\frac{\alpha}{2} & \pm\cos\frac{\alpha}{2} \end{pmatrix}, \quad 0 < \alpha < \pi.$$

Since the transformation $T_1 = D_2$ replaces $\sin\frac{\alpha}{2}$ by $-\sin\frac{\alpha}{2}$, we can write

$$N_2 = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}, \quad 0 < \alpha < \pi \tag{63}$$

(note that two last formulas for N_2 are not equivalent because (63) includes the extra value $\alpha = \pi/2$ corresponding to the case $N_2' = -I$).

Now we must prove the invariance of the parameter α. To this end suppose that a nondegenerate matrix T_1 satisfies (61), where N_2 has form (63) and

$$\widetilde{N_2} = \begin{pmatrix} \cos\tilde{\alpha} & \sin\tilde{\alpha} \\ -\sin\tilde{\alpha} & \cos\tilde{\alpha} \end{pmatrix}, \quad 0 < \tilde{\alpha} < \pi.$$

As $N_2 + N_2^* = T_1(\widetilde{N_2} + \widetilde{N_2}^*)T_1^*$ and $N_2 - N_2^* = T_1(\widetilde{N_2} - \widetilde{N_2}^*)T_1^*$, we have

$$4\cos^2\alpha = det(N_2 + N_2^*) = (detT_1)^2\, det(\widetilde{N_2} + \widetilde{N_2^*}) = (detT_1)^2 4\cos^2\tilde{\alpha}$$

and

$$4\sin^2\alpha = det(N_2 - N_2^*) = (detT_1)^2\, det(\widetilde{N_2} - \widetilde{N_2^*}) = (detT_1)^2 4\sin^2\tilde{\alpha}.$$

Therefore, $|detT_1| = 1$, $\cos\alpha = \pm\cos\tilde{\alpha}$, $\sin\alpha = \sin\tilde{\alpha}$. Now we write the condition $N_2 + N_2^* = T_1(\widetilde{N_2} + \widetilde{N_2}^*)T_1^*$ in detail:

$$\begin{pmatrix} \cos\alpha & 0 \\ 0 & \cos\alpha \end{pmatrix} = \begin{pmatrix} (t_{11}^2 + t_{12}^2)\cos\tilde{\alpha} & (t_{11}t_{21} + t_{12}t_{22})\cos\tilde{\alpha} \\ (t_{11}t_{21} + t_{12}t_{22})\cos\tilde{\alpha} & (t_{21}^2 + t_{22}^2)\cos\tilde{\alpha} \end{pmatrix}.$$

Since $|\cos\alpha| = |\cos\tilde{\alpha}|$, we have $t_{11}^2 + t_{12}^2 = 1$, hence $\cos\alpha = \cos\tilde{\alpha}$. Thus, $\alpha = \tilde{\alpha}$, Q.E.D.

If N_2' has distinct real eigenvalues r and $1/r$, i.e., $r \neq \pm 1$, then it can be reduced to the diagonal form $N_2' = 1/r \oplus r$, $|r| > 1$. Consequently,

$$N_2 = \begin{pmatrix} 0 & t \\ rt & 0 \end{pmatrix}, \ t \neq 0.$$

Taking $T_1 = 1 \oplus t$, we reduce N_2 to the form

$$N_2 = \begin{pmatrix} 0 & 1 \\ r & 0 \end{pmatrix}, \ |r| > 1. \tag{64}$$

It is clear that r is an invariant.

Finally, we consider the case when N_2' has the eigenvalues ± 1. If $N_2' = I$, the matrix N_2 is selfadjoint, hence, it can be reduced to the diagonal form. Therefore, the nondegenerate subspace $V = span\{v_1, v_3\}$ is invariant both for N and for $N^{[*]}$, i.e., the operator N is decomposable. It can easily be checked that N_2' is not equivalent to the form

$$N_2' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

because then N_2 turns out to be degenerate, which is impossible. If $N_2' = -I$, N_2 can be reduced to the above-mentioned form

$$N_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The last case to be considered is the case when the Jordan normal form of N_2' is

$$N_2' = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Then

$$N_2 = \begin{pmatrix} \frac{1}{2}t & t \\ -t & 0 \end{pmatrix}, \ t \neq 0.$$

Taking $T_1 = \sqrt{|t|}I$, we achieve

$$N_2 = \begin{pmatrix} \frac{1}{2}z & z \\ -z & 0 \end{pmatrix}, \ z = \pm 1. \tag{65}$$

Here z is an invariant. Indeed, suppose that some matrix T_1 satisfies condition (61), where

$$\widetilde{N_2} = \begin{pmatrix} \frac{1}{2}\tilde{z} & \tilde{z} \\ -\tilde{z} & 0 \end{pmatrix}, \ \tilde{z} = \pm 1.$$

Then $\frac{1}{2}z = \frac{1}{2}t_{11}^2\tilde{z}$, hence $z = \tilde{z}$.

As a result, we have obtained three forms (63), (64), (65). Now it is necessary to find out whether the operator N is indecomposable in the three cases. The indecomposability of N means that $(aN_2 + bN_2^*)x = 0$ only if $(x, N_2x) = 0$ $(a^2 + b^2 \neq 0)$. If $N_2' = N_2 N_2^{*-1}$ has no real eigenvalues, the equation $(aN_2 + bN_2^*)x = 0$ has no solutions, i.e., N is indecomposable if N_2 has form (63) with $\alpha \neq \pi/2$. If an eigenvalue λ of N_2' is not equal to 1, then $(x, N_2x) = 0$ because $(x, N_2x) = (x, \lambda N_2^* x) = \lambda(x, N_2^* x) = \lambda(x, N_2x)$. Thus, if N_2 has form (64), (65), or (63) with $\alpha = \pi/2$, then N is also indecomposable.

(b) $det N_2 = 0$. Since N with $N_2 = 0$ is decomposable, it suffices to consider the remaining case $rg\, N_2 = 1$:

$$N_2 = \begin{pmatrix} ka & kb \\ la & lb \end{pmatrix}, \ a^2 + b^2 \neq 0, \ k^2 + l^2 \neq 0.$$

It is readily seen that $S_0 \cap S_1 \neq \{0\}$ if $la = kb$, therefore, we can assume that this condition is not satisfied. Taking $T = T_1 \oplus T_1^{*-1}$, where

$$T_1 = \begin{pmatrix} a & k \\ b & l \end{pmatrix},$$

we obtain one more canonical form:

$$N - \lambda I = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(it can easily be checked that this form is indecomposable).

As a result, we have proved that

If an indecomposable H-normal operator N $(N : C^4 \to C^4)$ has the only eigenvalue $\lambda \in \Re$, and $dim\, S_0 = 2$, then the pair $\{N, H\}$ is unitarily similar to one and only one of the canonical pairs $\{(26),(30)\}$, $\{(27),(30)\}$, $\{(28),(30)\}$, $\{(29),(30)\}$.

4.2 Two Real Distinct Eigenvalues of N

Since the canonical pair $\{(42),(43)\}$ is obtained in the same way as in [2], we will not repeat the proof of the following fact:

If an indecomposable H-normal operator acts in a space R^n of rank 2 and has 2 distinct real eigenvalues: λ_1 and λ_2, then $n = 4$ and the pair $\{N, H\}$ is unitarily similar to the canonical pair $\{(42),(43)\}$.

4.3 Three Eigenvalues of N: One Real and Two Complex Conjugate

Suppose an indecomposable H-normal operator N has a real eigenvalue λ and two complex eigenvalues $\alpha \pm i\beta$ $(\beta > 0)$. According to Lemma 2.1, we have $R^n = \mathcal{Q}_1 \dot{+} \mathcal{Q}_2$, $dim\, \mathcal{Q}_1 = dim\, \mathcal{Q}_2 = m$, $[\mathcal{Q}_1, \mathcal{Q}_1] = 0$, $[\mathcal{Q}_2, \mathcal{Q}_2] = 0$, $N\mathcal{Q}_1 \subseteq \mathcal{Q}_1$, $N\mathcal{Q}_2 \subseteq \mathcal{Q}_2$, $N_1 = N|_{\mathcal{Q}_1}$ has two eigenvalues $\alpha \pm i\beta$, $N_2 = N|_{\mathcal{Q}_2}$ one eigenvalue λ. Since $min\{v_-, v_+\} = 2$, $n = v_- + v_+ \geq 4$. On the other hand, the subspaces $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are neutral so that $n = 2m \leq 4$. Thus, $n = 4$. As H is nondegenerate, for any basis in $\mathcal{Q}_1$ there exists a basis in $\mathcal{Q}_2$ such that

$$H = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Take a basis in $\mathcal{Q}_1$ such that

$$N_1 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}. \tag{66}$$

Then with respect to the decomposition $R^n = \mathcal{Q}_1 \dot{+} \mathcal{Q}_2$ we have

$$N = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{67}$$

The condition of the H-normality of N is

$$N_1 N_2^* = N_2^* N_1. \tag{68}$$

The only matrix commuting with (66) and having one eigenvalue λ is λI. Thus,

$$N = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \oplus \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

It can easily be checked that N is indecomposable. Indeed, suppose a subspace V is invariant for N and $N^{[*]}$. Since $min\{dim\, V, dim\, V^{[\perp]}\} \leq 2$, we can assume that $dim\, V \leq 2$. If V were of dimension 1, then there would exist a vector $v \in V$ such that $Nv = \lambda v$, $N^{[*]}v = \lambda v$. But all eigenvectors of N corresponding to the eigenvalue λ are not eigenvectors of $N^{[*]}$. Thus, $dim\, V \neq 1$. Suppose $dim\, V = 2$. Then $N|_V$ has either the only eigenvalue λ or two eigenvalues $\alpha \pm i\beta$. In the former case $V = \mathcal{Q}_2$, in the latter $V = \mathcal{Q}_1$. In the both cases V is degenerate, therefore, N is indecomposable.

Thus, we have proved that

If an indecomposable H-normal operator acts in a space R^n of rank 2 and has 3 eigenvalues: $\lambda \in R$, $\alpha \pm i\beta$ $(\alpha, \beta \in \Re,\ \beta > 0)$, then $n = 4$ and the pair $\{N, H\}$ is unitarily similar to the canonical pair $\{(44),(45)\}$.

4.4 Two Distinct Pairs of Complex Conjugate Eigenvalues of N

Suppose N has four eigevalues $\alpha_1 \pm i\beta_1$, $\alpha_2 \pm i\beta_2$ $(\beta_1,\ \beta_2 > 0,\ (\alpha_1, \beta_1) \neq (\alpha_2, \beta_2))$. Let us fix the order of these pairs: $\beta_1 \leq \beta_2$, $\alpha_1 < \alpha_2$ if $\beta_1 = \beta_2$. As in the previous section, one can show that N and H can be reduced to form (67) with

$$N_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{pmatrix}.$$

It follows from condition (68) of the H-normality of N that

$$N_2 = \begin{pmatrix} \alpha_2 & z\beta_2 \\ -z\beta_2 & \alpha_2 \end{pmatrix}, \quad z = \pm 1.$$

Now we prove that the number z is an H-unitary invariant. To this end suppose that a matrix T satisfies condition (9) $NT = T\tilde{N}$ and condition (10) $TT^{[*]} = I$, where

$$N = N_1 \oplus \begin{pmatrix} \alpha_2 & z\beta_2 \\ -z\beta_2 & \alpha_2 \end{pmatrix}, \quad \tilde{N} = N_1 \oplus \begin{pmatrix} \alpha_2 & \tilde{z}\beta_2 \\ -\tilde{z}\beta_2 & \alpha_2 \end{pmatrix}, \quad |z| = |\tilde{z}| = 1.$$

It follows from (9) that $T = T_1 \oplus T_2$, where

$$T_1 = \begin{pmatrix} t_{11} & t_{12} \\ -t_{12} & t_{11} \end{pmatrix}.$$

It follows from (10) that $T_2 = T_1^{*-1}$, therefore,

$$T_2 = \begin{pmatrix} t_{33} & t_{34} \\ -t_{34} & t_{33} \end{pmatrix}.$$

It is seen that under these conditions $\tilde{z} = z$. The indecomposability of the form obtained can be checked as before.

Thus, we have proved that

If an indecomposable H-normal operator acts in a space R^n of rank 2 and has 4 eigenvalues: $\alpha_1 \pm i\beta_1$, $\alpha_2 \pm i\beta_2$, $(\alpha_1, \beta_1, \alpha_2, \beta_2 \in \Re,\ 0 < \beta_1 \leq \beta_2,\ \alpha_1 < \alpha_2$ if $\beta_1 = \beta_2)$, then $n = 4$ and the pair $\{N, H\}$ is unitarily similar to the canonical pair $\{(46),(47)\}$.

4.5 Two Complex Conjugate Eigenvalues of N

The two following propositions hold for any space with indefinite scalar product. They are in a sense analogous to Propositions 1, 2 from [2].

Proposition 2 *Let an indecomposable H-normal operator N acting in R^n $(n > 2)$ have two distinct eigenvalues $\lambda = \alpha + i\beta$, $\overline{\lambda} = \alpha - i\beta$. Let*

$$S_0' = \{z = x + iy \ (x, y \in R^n): \ Nz = \lambda z, \ N^{[*]}z = \overline{\lambda}z\},$$

$$S_0'' = \{z = x + iy \ (x, y \in R^n): \ Nz = \lambda z, \ N^{[*]}z = \lambda z\},$$

$\{z_j\}_1^p$ $(\{z_j\}_{p+1}^{p+q})$ *be a basis of* S_0' (S_0''), *and*

$$S_0 = \sum_{j=1}^{p+q} span\{x_j, y_j\}.$$

Then there exists a decomposition of R^n into a direct sum of subspaces S_0, S, S_1 such that

$$N = \begin{pmatrix} N' & * & * \\ 0 & N_1 & * \\ 0 & 0 & N'' \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 & I \\ 0 & H_1 & 0 \\ I & 0 & 0 \end{pmatrix}, \tag{69}$$

where

$$N': \ S_0 \to S_0, \quad N' = N_1' \oplus \ldots \oplus N_{p+q}',$$

$$N_j' = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \quad j = 1, \ldots p+q, \tag{70}$$

$$N'': \ S_1 \to S_1, \quad N'' = N_1'' \oplus \ldots \oplus N_{p+q}'',$$

$$N_j'' = N_j' \ \ if \ 1 \le j \le p, \quad N_j'' = N_j'^{*} \ \ if \ p < j \le p+q, \tag{71}$$

the internal operator N_1 is H_1-normal and the pair $\{N_1, H_1\}$ is determined up to unitarily similarity. To go over from one decomposition $R^n = S_0 \dot{+} S \dot{+} S_1$ to another by means of a transformation T it is necessary that the matrix T be block triangular with respect to both decompositions.

Proof: It is clear that the subspace S_0 is well defined, i.e., that its definition does not depend on the choice of bases in S_0' and S_0''. Since N and $N^{[*]}$ commute and have two eigenvalues, at least one of the subspaces S_0', S_0'' is nontrivial so that $p + q > 0$. Show that the system $\{x_j\}_1^{p+q} \cup \{y_j\}_1^{p+q}$ is a basis in S_0. In fact, the assumption $\sum_{j=1}^{p+q}(a_j x_j + b_j y_j) = 0$ $(a_j, b_j \in \Re, j = 1, \ldots p+q)$ means that $\mathcal{R}e \sum_{j=1}^{p+q}(a_j - ib_j)z_j = 0$, therefore, $\mathcal{R}e\{N \sum_{j=1}^{p+q}(a_j - ib_j)z_j\} = 0$. But $\mathcal{R}e\{N \sum_{j=1}^{p+q}(a_j - ib_j)z_j\} = \alpha \mathcal{R}e \sum_{j=1}^{p+q}(a_j - ib_j)z_j - \beta \mathcal{I}m \sum_{j=1}^{p+q}(a_j - ib_j)z_j$ so that $\mathcal{I}m \sum_{j=1}^{p+q}(a_j - ib_j)z_j = 0$. Thus, $\sum_{j=1}^{p+q}(a_j - ib_j)z_j = 0$. Since the vectors z_j are linearly independent in C^n, $a_j = b_j = 0$ $(j = 1, \ldots p+q)$, i.e., the vectors $\{x_j\}_1^{p+q} \cup \{y_j\}_1^{p+q}$ are linearly independent in R^n. Thus, the dimension of S_0 is equal to $2(p+q)$.

Now let us prove that for N to be indecomposable it is necesssary that S_0 be neutral. Indeed, we already know that if $z = x + iy$ $(x, y \in R^n)$ is an eigenvector of $N^{[*]}$ such that $Nz = \lambda z$, then the subspace $span\{x, y\}$, which is invariant for N and $N^{[*]}$, is either nondegenerate or neutral (see Section 2.3). Since $n > 2$ and N is indecomposable, it is necessarily neutral. Further, if $Nz_1 = \lambda z_1$, $N^{[*]}z_1 = \overline{\lambda}z_1$, $Nz_2 = \lambda z_2$, $N^{[*]}z_2 = \lambda z_2$, then it can be shown (as in Section 2.3) that $[z_1, z_2] = [z_1, \overline{z_2}] = 0$, hence $[x_1, x_2] = [x_1, y_2] = [y_1, x_2] = [y_1, y_2] = 0$. If $Nz_1 = \lambda z_1$, $N^{[*]}z_1 = \overline{\lambda}z_1$, $Nz_2 = \lambda z_2$, $N^{[*]}z_2 = \overline{\lambda}z_2$, then $[z_1, \overline{z_2}] = 0$, i.e., $[x_1, x_2] = [y_1, y_2]$ and $[x_1, y_2] = -[y_1, x_2]$. If $a^2 + b^2 \neq 0$ $(a = [x_1, x_2]$, $b = [x_1, y_2])$, the two-dimensional subspace $span\{ax_1 - by_1 + x_2, bx_1 + ay_1 + y_2\}$, which is invariant for N and $N^{[*]}$, will be nondegenerate, therefore, N will be decomposable. Thus, for N to be indecomposable it is necessary to have $a = b = 0$. It can be checked in the similar way that the conditions $[x_1, x_2] = [y_1, y_2] = [x_1, y_2] = [y_1, x_2] = 0$

are satisfied if $Nz_1 = \lambda z_1$, $N^{[*]}z_1 = \lambda z_1$, $Nz_2 = \lambda z_2$, $N^{[*]}z_2 = \lambda z_2$. Thus, if N is indecomposable, S_0 is neutral.

For any neutral subspace S_0 of a space with indefinite scalar product there exists a subspace S_1 such that

$$H|_{(S_0 \dot{+} S_1)} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Since $(S_0 \dot{+} S_1)$ is nondegenerate, the subspace $S = (S_0 \dot{+} S_1)^{[\perp]}$ is nondegenerate too and $R^n = S_0 \dot{+} S \dot{+} S_1$. It is clear that with respect to this decomposition the matrices N and H have form (69), the submatrix N' has form (70) and N'' has from (71). The last two statements of the proposition can be proved as in Proposition 1 from [2]. The proof is completed.

Proposition 3 *An H-normal operator such that $dim\ S_0 = 2$ is indecomposable.*

Proof: Assume the converse. Suppose some nondegenerate subspace V is invariant both for N and for $N^{[*]}$. Let us denote $V_1 = V$, $V_2 = V^{[\perp]}$, $N_1 = N|_{V_1}$, $N_2 = N|_{V_2}$, $H_1 = H|_{V_1}$, $H_2 = H|_{V_2}$. Since the operators N_i $(i = 1, 2)$ are H_i-normal, both subspaces $S_0^{(i)} \subset V_i$ (defined as S_0) are nontrivial, i.e., $dim\ S_0^{(i)} \geq 2$. Since $S_0 = S_0^{(1)} \dot{+} S_0^{(2)}$, $dim\ S_0 = dim\ S_0^{(1)} + dim\ S_0^{(2)} \geq 4$. This contradicts the condition $dim\ S_0 = 2$. Thus, N is indecomposable.

Now let us show that if $min\{v_-, v_+\} = 2$, then N is indecomposable only if $n \leq 8$. According to Proposition 2, which is applicable (recall that $n = v_- + v_+ \geq 4$), if N is indecomposable, then S_0 is neutral so that $dim\ S_0 = 2$. Therefore, if we show that for $n > 8$ we have $dim\ S_0 > 2$, this will mean that N is decomposable.

Let us complexify the source space R^n and apply the results from [1] and [2] concerning the decomposition of an H-normal operator in a complex space. Lemma 1 from [1] states that for an H-normal operator having two distinct eigenvalues λ and $\overline{\lambda}$ there exists a decomposition of C^n into a sum $C^n = V_1 \dot{+} V_2 \dot{+} V_3 \dot{+} V_4$ such that

$$N = \begin{pmatrix} N_1 & 0 & 0 & 0 \\ 0 & N_2 & 0 & 0 \\ 0 & 0 & N_3 & 0 \\ 0 & 0 & 0 & N_4 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & H_3 & 0 \\ 0 & 0 & 0 & H_4 \end{pmatrix},$$

where N_1, N_3 have the only eigenvalue λ, N_2, N_4 the only eigenvalue $\overline{\lambda}$, $dimV_1 = dimV_2$. It is seen that if the space C^n is R^n complexified, then $dimV_3 = dimV_4$.

Since ranks of the subspaces $V_1 \dot{+} V_2$, V_3, V_4 are less than or equal to 2, Theorem 1 from [1] and Theorem 1 from [2] are applicable. It follows from these theorems that if $dim\ V_1,\ dim\ V_3 > 0$, then there exist at least two linearly independent vectors z_1, z_2 such that $Nz_1 = \lambda z_1$, $N^{[*]}z_1 = \lambda z_1$, $Nz_2 = \lambda z_2$, $N^{[*]}z_2 = \overline{\lambda} z_2$, i.e., $dimS_0 \geq 4$. If $dimV_3 = 0$, n is equal to 4 because the subspaces V_1 and V_2 are neutral (hence $n = (2\, dimV_1) \leq 4 \Rightarrow n = 4$). If $dimV_1 = 0$, there appear two alternatives: V_3 and V_4 each have rank 1 or one of these subspaces has rank 0. In the latter case either N_3 or N_4 is decomposable for any n. In the former case, according to Theorem 1 [1], N_3 (N_4) is always decomposable if $dimV_3 > 4$ ($dimV_4 > 4$). In either case for $n > 8$ there exist two linearly independent vectors z_1, z_2 such that $Nz_1 = \lambda z_1$, $N^{[*]}z_1 = \overline{\lambda} z_1$, $Nz_2 = \lambda z_2$, $N^{[*]}z_2 = \overline{\lambda} z_2$. As above, we have $dim\ S_0 \geq 4$. Thus, if $n > 8$, N is decomposable, Q.E.D.

Thus, according to Proposition 2, the matrices N and H can be reduced to the form

$$N = \begin{pmatrix} N_1 & N_2 & N_3 \\ 0 & N_4 & N_5 \\ 0 & 0 & N_6 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 & I \\ 0 & I & 0 \\ I & 0 & 0 \end{pmatrix}, \tag{72}$$

where

$$N_1 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},$$

N_6 is equal either to N_1 or to N_1^*. The condition of the H-normality of N is equivalent to the system

$$\begin{aligned} N_1 N_6^* &= N_6^* N_1, & (73)\\ N_1 N_5^* + N_2 N_4^* &= N_6^* N_2 + N_5^* N_4, & (74)\\ N_1 N_3^* + N_2 N_2^* + N_3 N_1^* &= N_6^* N_3 + N_5^* N_5 + N_3^* N_6, & (75)\\ N_4 N_4^* &= N_4^* N_4. & (76) \end{aligned}$$

Note that if $N_6 = N_1^*$, then $dim\ S_0'' > 0$ so that it is the case $dim V_1 > 0$. It was stated before that if $dim V_1 > 0$, then for indecomposable operators $n = 4$. Therefore, for $n = 4$ the submatrix N_6 can be equal to either N_1 or N_1^* but for $n = 6, 8$ we have $N_6 = N_1$. Now let us consider the cases $n = 4, 6, 8$ successively.

4.5.1 $n = 4$

By the above,

$$N = \begin{pmatrix} N_1 & N_3 \\ 0 & N_6 \end{pmatrix} = \begin{pmatrix} \alpha & \beta & a & b \\ -\beta & \alpha & c & d \\ 0 & 0 & \alpha & \pm\beta \\ 0 & 0 & \mp\beta & \alpha \end{pmatrix}.$$

$\underline{N_6 = N_1}$ Then from (75) it follows that $c = -b$, $d = a$. If $a^2 + b^2 = 0$, i.e., $N_3 = 0$, then $S_0 \cap S_1 \neq 0$, which contradicts the indecomposability of N. Therefore, $a^2 + b^2 \neq 0$. Taking the block diagonal transformation $T = \sqrt[4]{a^2+b^2} I_2 \oplus 1/\sqrt[4]{a^2+b^2} I_2$, we can reduce N to the form

$$N = \begin{pmatrix} \alpha & \beta & \cos\gamma & \sin\gamma \\ -\beta & \alpha & -\sin\gamma & \cos\gamma \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta & \alpha \end{pmatrix}, \quad 0 \leq \gamma < 2\pi. \tag{77}$$

According to Proposition 3, matrix (77) is indecomposable. Let us prove the H-unitary invariance of the parameter γ. To this end suppose that a matrix T satisfies conditions

$$\begin{aligned} NT &= T\tilde{N}, & (78)\\ TT^{[*]} &= I & (79) \end{aligned}$$

for the matrix N of form (77) and the matrix

$$\tilde{N} = \begin{pmatrix} N_1 & \widetilde{N_3} \\ 0 & N_1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \cos\tilde{\gamma} & \sin\tilde{\gamma} \\ -\beta & \alpha & -\sin\tilde{\gamma} & \cos\tilde{\gamma} \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta & \alpha \end{pmatrix}, \quad 0 \leq \tilde{\gamma} < 2\pi.$$

According to Proposition 2, the matrix T has the block triangular form

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

with respect to the decomposition $R^4 = S_0 \dot{+} S_1$. The transformation T is H-unitary iff

$$\begin{aligned} T_1 T_3^* &= I, & (80)\\ T_1 T_2^* + T_2 T_1^* &= 0. & (81) \end{aligned}$$

It follows from condition (78) that N_1 and T_1 commute, therefore,

$$T_1 = \begin{pmatrix} t_{11} & t_{12} \\ -t_{12} & t_{11} \end{pmatrix}$$

so that from (81) we get

$$T_2 = \begin{pmatrix} t_{13} & t_{14} \\ -t_{14} & t_{13} \end{pmatrix}.$$

Now, combining (80) and (78), we have $N_1 T_2 + N_3 T_1^{*-1} = T_1 \widetilde{N_3} + T_2 N_1$. But T_2 and N_1 commute (as well as T_1 and $\widetilde{N_3}$) so that $N_3 = T_1 \widetilde{N_3} T_1^* = \widetilde{N_3} T_1 T_1^* = (det\, T_1)^2 \widetilde{N_3}$. Since $det N_3 = det \widetilde{N_3} = 1$, we have $(det T_1)^2 = 1$ and $N_3 = \widetilde{N_3}$, i.e., $\gamma = \tilde{\gamma}$, Q.E.D.

$\underline{N_6 = N_1^*}$ Then, according to (75), $c = b$. The transformation

$$T = \begin{pmatrix} 1 & 0 & 0 & a/2\beta \\ 0 & 1 & -a/2\beta & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

reduces N_3 to the form

$$N_3 = \begin{pmatrix} 0 & b' \\ b' & d' \end{pmatrix}$$

without changing the submatrices N_1 and N_6. If both b' and d' are equal to zero, the condition $S_0 \cap S_1 = \{0\}$ fails. Therefore, $4b'^2 + d'^2 \neq 0$ and we can take the transformation

$$T = \begin{pmatrix} \cos\phi & \sin\phi & -r\sin\phi & r\cos\phi \\ -\sin\phi & \cos\phi & -r\cos\phi & -r\sin\phi \\ 0 & 0 & \cos\phi & \sin\phi \\ 0 & 0 & -\sin\phi & \cos\phi \end{pmatrix},$$

where $\cos 2\phi = 2b'/\sqrt{d'^2 + 4b'^2}$, $\sin 2\phi = -d'/\sqrt{d'^2 + 4b'^2}$, $r = d'/(4\beta)$. It does not change N_1 and N_6 but reduces N_3 to the form

$$N_3 = \begin{pmatrix} 0 & b'' \\ b'' & 0 \end{pmatrix}, \; b'' = \frac{1}{2}\sqrt{4b'^2 + d'^2} > 0.$$

If we now take $\widetilde{v_1} = \sqrt{b''} v_1$, $\widetilde{v_2} = \sqrt{b''} v_2$, $\widetilde{v_3} = v_3/\sqrt{b''}$, $\widetilde{v_4} = v_4/\sqrt{b''}$, then N_3 will be equal to D_2. Thus, we have obtained the final form for the matrix N:

$$N = \begin{pmatrix} \alpha & \beta & 0 & 1 \\ -\beta & \alpha & 1 & 0 \\ 0 & 0 & \alpha & -\beta \\ 0 & 0 & \beta & \alpha \end{pmatrix}. \tag{82}$$

According to Proposition 3, matrix (82) is indecomposable. Forms (77) and (82) are not H-unitarily similar because for matrix (82) the subspace S_0'' defined in Proposition 2 is nontrivial in contrast to that for (77). Thus, we have proved that

If an indecomposable H-normal operator acts in a space R^4 of rank 2 and has 2 eigenvalues: $\alpha \pm i\beta$ $(\alpha, \beta \in \Re,\ \beta > 0)$, then the pair $\{N, H\}$ is unitarily similar to one and only one of the canonical pairs $\{(48),(50)\}$, $\{(49),(50)\}$.

4.5.2 $n = 6$

The matrices N and H have form (72) with $N_6 = N_1$. Since the submatrix N_4 is an ordinary normal matrix (condition (76)), one can assume that $N_4 = N_1$. Thus,

$$N = \begin{pmatrix} N_1 & N_2 & N_3 \\ 0 & N_1 & N_5 \\ 0 & 0 & N_1 \end{pmatrix}.$$

First reduce the submatrix

$$N_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

to the form

$$N_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{83}$$

without changing the submatrices $N_1 = N_4 = N_6$. To this end take

$$T = \begin{pmatrix} I & T_2 & -\frac{1}{2}T_2T_2^* \\ 0 & I & -T_2^* \\ 0 & 0 & I \end{pmatrix}, \tag{84}$$

$$T_2 = \begin{pmatrix} b/\beta & -a/\beta \\ 0 & 0 \end{pmatrix}.$$

Then

$$N_2 = \begin{pmatrix} 0 & 0 \\ c' & d' \end{pmatrix}.$$

If both c' and d' are equal to zero, i.e., $N_2 = 0$, then from condition of the H-normality (75) it follows that $N_5 = 0$, which contradicts the condition $S_0 \cap S = \{0\}$. Therefore, $c'^2 + d'^2 \neq 0$ and we can subject the matrix N obtained to the transformation $T = I_2 \oplus T_1 \oplus I_2$, where

$$T_1 = \begin{pmatrix} d'/\sqrt{c'^2+d'^2} & c'/\sqrt{c'^2+d'^2} \\ -c'/\sqrt{c'^2+d'^2} & d'/\sqrt{c'^2+d'^2} \end{pmatrix}.$$

Then

$$N_2 = \begin{pmatrix} 0 & 0 \\ 0 & d'' \end{pmatrix}, \quad d'' = \sqrt{c'^2+d'^2} > 0.$$

Taking $\widetilde{v_1} = d''v_1$, $\widetilde{v_2} = d''v_2$, $\widetilde{v_3} = v_3$, $\widetilde{v_4} = v_4$, $\widetilde{v_5} = v_5/d''$, $\widetilde{v_6} = v_6/d''$, we obtain desired form (83) for the submatrix N_2.

Now let us apply conditions (74) and (75). We get

$$N_5 = \frac{1}{2}\begin{pmatrix} \cos\gamma + 1 & \sin\gamma \\ -\sin\gamma & \cos\gamma - 1 \end{pmatrix}, \quad 0 \le \gamma < 2\pi,$$

$$N_3 = \begin{pmatrix} p & q \\ (\cos\gamma+1)/4\beta - q & \sin\gamma/4\beta + p \end{pmatrix}.$$

Finally, take transformation (84) with

$$T_2 = 2p/(\cos\gamma+1)\, I_2 \;\text{ if }\; \gamma \neq \pi,$$

$$T_2 = \begin{pmatrix} 0 & -q \\ q & 0 \end{pmatrix} \;\text{ if }\; \gamma = \pi.$$

Then

$$N_3 = \begin{pmatrix} 0 & q' \\ (\cos\gamma+1)/4\beta - q' & \sin\gamma/4\beta \end{pmatrix} \;(\gamma \neq \pi),$$

$$N_3 = p' I_2 \;(\gamma = \pi).$$

As a result, we have obtained two forms:

$$N = \begin{pmatrix} \alpha & \beta & 0 & 0 & 0 & r \\ -\beta & \alpha & 0 & 1 & (\cos\gamma+1)/4\beta - r & \sin\gamma/4\beta \\ 0 & 0 & \alpha & \beta & \frac{1}{2}(\cos\gamma+1) & \frac{1}{2}\sin\gamma \\ 0 & 0 & -\beta & \alpha & -\frac{1}{2}\sin\gamma & \frac{1}{2}(\cos\gamma-1) \\ 0 & 0 & 0 & 0 & \alpha & \beta \\ 0 & 0 & 0 & 0 & -\beta & \alpha \end{pmatrix},$$

$$0 \le \gamma < 2\pi,\ \gamma \neq \pi, \tag{85}$$

$$N = \begin{pmatrix} \alpha & \beta & 0 & 0 & r & 0 \\ -\beta & \alpha & 0 & 1 & 0 & r \\ 0 & 0 & \alpha & \beta & 0 & 0 \\ 0 & 0 & -\beta & \alpha & 0 & -1 \\ 0 & 0 & 0 & 0 & \alpha & \beta \\ 0 & 0 & 0 & 0 & -\beta & \alpha \end{pmatrix}. \tag{86}$$

According to Proposition 3, matrices (85) and (86) are indecomposable. Let us show that they are not H-unitarily similar and that the numbers r and γ are H-unitary invariants. To this end suppose that some H-unitary matrix T reduces the matrix N to the form $\tilde{N}$:

$$N = \begin{pmatrix} N_1 & N_2 & N_3 \\ 0 & N_1 & N_5 \\ 0 & 0 & N_1 \end{pmatrix},\ \tilde{N} = \begin{pmatrix} N_1 & N_2 & \widetilde{N_3} \\ 0 & N_1 & \widetilde{N_5} \\ 0 & 0 & N_1 \end{pmatrix},$$

where

$$N_1 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},\ N_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$N_5 = \frac{1}{2}\begin{pmatrix} \cos\gamma + 1 & \sin\gamma \\ -\sin\gamma & \cos\gamma - 1 \end{pmatrix},\ 0 \le \gamma < 2\pi,$$

$$\widetilde{N_5} = \frac{1}{2}\begin{pmatrix} \cos\tilde{\gamma} + 1 & \sin\tilde{\gamma} \\ -\sin\tilde{\gamma} & \cos\tilde{\gamma} - 1 \end{pmatrix},\ 0 \le \tilde{\gamma} < 2\pi.$$

Then, according to Proposition 2, T has the block triangular form

$$T = \begin{pmatrix} T_1 & T_2 & T_3 \\ 0 & T_4 & T_5 \\ 0 & 0 & T_6 \end{pmatrix}$$

with respect to the decomposition $R^6 = S_0 \dot{+} S \dot{+} S_1$. It follows from condition (78) $NT = T\tilde{N}$ that

$$T_1 = T_4 = T_6 = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix},\ T_2 = \begin{pmatrix} t_{13} & t_{14} \\ -t_{14} & t_{13} + \frac{\sin\phi}{\beta} \end{pmatrix}.$$

Condition (79) $TT^{[*]} = I$ implies $T_1T_5^* + T_2T_4^* = 0$, hence

$$T_5 = -T_1T_2^*T_1 = \begin{pmatrix} t_{35} & t_{36} \\ -t_{36} & t_{35} - \frac{\sin\phi}{\beta} \end{pmatrix},$$

where

$$\begin{aligned} t_{35} &= -t_{13}\cos 2\phi - t_{14}\sin 2\phi + \frac{\sin^3\phi}{\beta}, \\ t_{36} &= -t_{13}\sin 2\phi + t_{14}\cos 2\phi - \frac{\cos\phi\sin^2\phi}{\beta}. \end{aligned}$$

Substituting the expressions for T_4, T_5, T_6 in the formula $N_1T_5 + N_5T_6 = T_4\widetilde{N_5} + T_5N_1$, which follows from (78), we obtain: $N_5 = \widetilde{N_5}$. Therefore, forms (85) and (86) are not H-unitarily similar and the parameter γ is an H-unitary invariant.

Now let us check the H-unitary invariance of r for matrix (85). To this end suppose that

$$N_3 = \begin{pmatrix} 0 & r \\ (\cos\gamma + 1)/4\beta - r & \sin\gamma/4\beta \end{pmatrix},$$

$$\widetilde{N_3} = \begin{pmatrix} 0 & \tilde{r} \\ (\cos\gamma+1)/4\beta - \tilde{r} & \sin\gamma/4\beta \end{pmatrix},$$

$0 \le \gamma < 2\pi$, $\gamma \ne \pi$. It follows from (79) that $T_1T_3^* = -\frac{1}{2}T_2T_2^* + X$, where X is an antisymmetric matrix, therefore,

$$T_3 = \begin{pmatrix} t_{15} & t_{16} \\ t_{25} & t_{26} \end{pmatrix} - \begin{pmatrix} x\sin\phi & -x\cos\phi \\ x\cos\phi & x\sin\phi \end{pmatrix},$$

where

$$\begin{aligned} 2t_{15} &= -(t_{13}^2 + t_{14}^2)\cos\phi + t_{14}\sin^2\phi/\beta, \\ 2t_{16} &= -(t_{13}^2 + t_{14}^2)\sin\phi - t_{14}\sin\phi\cos\phi/\beta, \\ 2t_{25} &= -t_{14}\sin\phi\cos\phi/\beta + ((t_{13} + \sin\phi/\beta)^2 + t_{14}^2)\sin\phi, \\ 2t_{26} &= -t_{14}\sin^2\phi/\beta - ((t_{13} + \sin\phi/\beta)^2 + t_{14}^2)\cos\phi. \end{aligned}$$

Since $N_1T_3 + N_2T_5 + N_3T_6 = T_1\widetilde{N_3} + T_2N_5 + T_3N_1$ (condition (78)), $\widetilde{N_3} = T_1^*(N_1T_3 - T_3N_1 + N_2T_5 - T_2N_5 + N_3T_6)$. Substituting the expressions for T_2, T_3, T_5, T_6 in this formula, we obtain:

$$a_1t_{13} + a_2t_{14} + a_3 = 0, \tag{87}$$
$$b_1t_{13} + b_2t_{14} + b_3 = \tilde{r} - r, \tag{88}$$

where

$$\begin{aligned} a_1 &= -\frac{1}{2}(\cos(\phi-\gamma) + \cos\phi), \\ a_2 &= -\frac{1}{2}(\sin(\phi-\gamma) + \sin\phi), \\ a_3 &= -\frac{1}{4\beta}\sin\phi(\cos(\phi-\gamma) + \cos\phi), \\ b_1 &= \frac{1}{2}(\sin(\phi-\gamma) - \sin\phi), \\ b_2 &= -\frac{1}{2}(\cos(\phi-\gamma) - \cos\phi), \\ b_3 &= \frac{1}{4\beta}\sin\phi(\sin(\phi-\gamma) - \sin\phi). \end{aligned}$$

Since the left hand sides of equations (87) - (88) are proportional and the coefficients of t_{13} and of t_{14} in (87) are not equal to zero simultaneously, condition (87) implies $\tilde{r} = r$. Therefore, r is an H-unitary invariant. The proof of the invariance of r for matrix (86) is analogous.

Thus, we have proved that

If an indecomposable H-normal operator acts in a space R^6 of rank 2 and has 2 eigenvalues: $\alpha \pm i\beta$ ($\alpha, \beta \in \Re$, $\beta > 0$), then the pair $\{N, H\}$ is unitarily similar to one and only one of the canonical pairs $\{$(51),(53)$\}$, $\{$(52),(53)$\}$.

4.5.3 $n = 8$

The matrices N and H have form (72), N_6 being equal to N_1:

$$N = \begin{pmatrix} N_1 & N_2 & N_3 \\ 0 & N_4 & N_5 \\ 0 & 0 & N_1 \end{pmatrix}.$$

Since N_4 is an ordinary normal matrix (condition (76)), it can be assumed that $N_4 = N_1 \oplus N_1$.

Having these equalities in mind, we reduce the submatrix

$$N_2 = \begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix}$$

to the form

$$N_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{89}$$

without changing the submatrices N_1, N_4, and $N_6 = N_1$. To this end take transformation (84) with

$$T_2 = \begin{pmatrix} b/\beta & -a/\beta & d/\beta & -c/\beta \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$N_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ e' & f' & g' & h' \end{pmatrix}.$$

Now subject the obtained matrix N to the transformation $T = I_2 \oplus T_1' \oplus T_1'' \oplus I_2$, where

$$T_1' = \begin{pmatrix} f'/\sqrt{e'^2+f'^2} & e'/\sqrt{e'^2+f'^2} \\ -e'/\sqrt{e'^2+f'^2} & f'/\sqrt{e'^2+f'^2} \end{pmatrix} \text{ if } e'^2+f'^2>0,$$

$$T_1' = I_2 \text{ if } e' = f' = 0,$$

$$T_1'' = \begin{pmatrix} h'/\sqrt{g'^2+h'^2} & g'/\sqrt{g'^2+h'^2} \\ -g'/\sqrt{g'^2+h'^2} & h'/\sqrt{g'^2+h'^2} \end{pmatrix} \text{ if } g'^2+h'^2>0,$$

$$T_1'' = I_2 \text{ if } g' = h' = 0.$$

We get

$$N_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & f'' & 0 & h'' \end{pmatrix},\ f'' = \sqrt{e'^2+f'^2} \geq 0,\ h'' = \sqrt{g'^2+h'^2} \geq 0.$$

If $f''+h''=0$, i.e., $N_2=0$, from condition (75) it follows that $N_5=0$, which is impossible because $S_0 \cap S = \{0\}$. Therefore, $f''+h''>0$. Without loss of generality it can be assumed that $f'' \neq 0$ (otherwise one can take $\widetilde{v_3} = v_5$, $\widetilde{v_4} = v_6$, $\widetilde{v_5} = v_3$, $\widetilde{v_6} = v_4$). Therefore, we can assume $f''=1$, taking $\widetilde{v_1} = f''v_1$, $\widetilde{v_2} = f''v_2$, $\widetilde{v_7} = v_7/f''$, $\widetilde{v_8} = v_8/f''$. Keeping in mind that $f''=1$, take the transformation

$$T = T_1 \oplus \begin{pmatrix} 1/\sqrt{1+h''^2}I_2 & -h''/\sqrt{1+h''^2}I_2 \\ h''/\sqrt{1+h''^2}I_2 & 1/\sqrt{1+h''^2}I_2 \end{pmatrix} \oplus T_1^{*-1},$$

where $T_1 = \sqrt{1+h''^2}I_2$. Then we obtain desired form (89) for the submatrix N_2.

Condition (74) implies

$$N_5 = \begin{pmatrix} * & * \\ * & * \\ p & q \\ -q & p \end{pmatrix}.$$

Since the case $p=q=0$ is impossible (the condition $S_0 \cap S = \{0\}$), we have $p^2+q^2>0$. The transformation $T = I_2 \oplus I_2 \oplus T_1 \oplus I_2$, where

$$T_1 = \begin{pmatrix} p/\sqrt{p^2+q^2} & q/\sqrt{p^2+q^2} \\ -q/\sqrt{p^2+q^2} & p/\sqrt{p^2+q^2} \end{pmatrix},$$

reduces N_5 to the form

$$N_5 = \begin{pmatrix} * & * \\ * & * \\ p' & 0 \\ 0 & p' \end{pmatrix},\ p' = \sqrt{p^2+q^2} > 0,$$

retaining the submatrices N_1, N_2, N_4, and N_6. It follows from conditions of the H-normality (74) and (75) that

$$N_5 = \begin{pmatrix} \sin^2\gamma & \sin\gamma\cos\gamma\cos\delta \\ -\sin\gamma\cos\gamma\cos\delta & -\cos^2\gamma \\ \sin\gamma\cos\gamma\sin\delta & 0 \\ 0 & \sin\gamma\cos\gamma\sin\delta \end{pmatrix},\ 0<\gamma<\pi/2,\ 0<\delta<\pi,$$

$$N_3 = \begin{pmatrix} s & t \\ \sin^2\gamma/2\beta - t & \sin\gamma\cos\gamma\cos\delta/2\beta + s \end{pmatrix}.$$

At last, take transformation (84), where

$$T_2 = \begin{pmatrix} 0 & 0 & s/(\sin\gamma\cos\gamma\sin\delta) & t/(\sin\gamma\cos\gamma\sin\delta) \\ 0 & 0 & -t/(\sin\gamma\cos\gamma\sin\delta) & s/(\sin\gamma\cos\gamma\sin\delta) \end{pmatrix},$$

and reduce N to the final form:

$$N = \begin{pmatrix} \alpha & \beta & 0 & 0 & 0 & 0 & 0 & 0 \\ -\beta & \alpha & 0 & 1 & 0 & 0 & \sin^2\gamma/2\beta & \sin\gamma\cos\gamma\cos\delta/2\beta \\ 0 & 0 & \alpha & \beta & 0 & 0 & \sin^2\gamma & \sin\gamma\cos\gamma\cos\delta \\ 0 & 0 & -\beta & \alpha & 0 & 0 & -\sin\gamma\cos\gamma\cos\delta & -\cos^2\gamma \\ 0 & 0 & 0 & 0 & \alpha & \beta & \sin\gamma\cos\gamma\sin\delta & 0 \\ 0 & 0 & 0 & 0 & -\beta & \alpha & 0 & \sin\gamma\cos\gamma\sin\delta \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha & \beta \\ 0 & 0 & 0 & 0 & 0 & 0 & -\beta & \alpha \end{pmatrix}$$

$$0<\gamma<\pi/2,\ 0<\delta<\pi. \tag{90}$$

Due to Proposition 3 the matrix obtained is indecomposable. Let us check the H-unitary invariance of the parameters γ and δ. Suppose some H-unitary matrix T reduces the matrix N to the form $\tilde{N}$:

$$N = \begin{pmatrix} N_1 & N_2 & N_3 \\ 0 & N_4 & N_5 \\ 0 & 0 & N_1 \end{pmatrix},\quad \tilde{N} = \begin{pmatrix} N_1 & N_2 & \widetilde{N_3} \\ 0 & N_4 & \widetilde{N_5} \\ 0 & 0 & N_1 \end{pmatrix},$$

where

$$N_1 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},\quad N_4 = N_1 \oplus N_1,\quad N_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$N_5 = \begin{pmatrix} \sin^2\gamma & \sin\gamma\cos\gamma\cos\delta \\ -\sin\gamma\cos\gamma\cos\delta & -\cos^2\gamma \\ \sin\gamma\cos\gamma\sin\delta & 0 \\ 0 & \sin\gamma\cos\gamma\sin\delta \end{pmatrix},$$

$$\widetilde{N_5} = \begin{pmatrix} \sin^2\tilde\gamma & \sin\tilde\gamma\cos\tilde\gamma\cos\tilde\delta \\ -\sin\tilde\gamma\cos\tilde\gamma\cos\tilde\delta & -\cos^2\tilde\gamma \\ \sin\tilde\gamma\cos\tilde\gamma\sin\tilde\delta & 0 \\ 0 & \sin\tilde\gamma\cos\tilde\gamma\sin\tilde\delta \end{pmatrix},$$

$$0<\gamma,\tilde\gamma<\pi/2,\ 0<\delta,\tilde\delta<\pi.$$

Then, according to Proposition 2, T has the block triangular form

$$T = \begin{pmatrix} T_1 & T_2 & T_3 \\ 0 & T_4 & T_5 \\ 0 & 0 & T_6 \end{pmatrix}$$

with respect to the decomposition $R^8 = S_0 \dot{+} S \dot{+} S_1$. Combining condition (78) $NT = T\tilde{N}$ and (79) $TT^{[*]} = I$, we get

$$T_1 = T_6 = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}, \quad T_4 = T_1 \oplus \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix},$$

$$T_2 = \begin{pmatrix} t_{13} & t_{14} & t_{15} & t_{16} \\ -t_{14} & t_{13} + \frac{\sin\phi}{\beta} & -t_{16} & t_{15} \end{pmatrix}, \quad T_5 = \begin{pmatrix} t_{37} & t_{38} \\ -t_{38} & t_{37} - \frac{\sin\phi}{\beta} \\ t_{57} & t_{58} \\ -t_{58} & t_{57} \end{pmatrix},$$

where

$$\begin{aligned} t_{37} &= -t_{13}\cos 2\phi - t_{14}\sin 2\phi + \frac{\sin^3\phi}{\beta}, \\ t_{38} &= -t_{13}\sin 2\phi + t_{14}\cos 2\phi - \frac{\cos\phi\sin^2\phi}{\beta}, \\ t_{57} &= -t_{15}\cos(\phi+\psi) - t_{16}\sin(\phi+\psi), \\ t_{58} &= -t_{15}\sin(\phi+\psi) + t_{16}\cos(\phi+\psi). \end{aligned}$$

Substituting the expressions for T_4, T_5, T_6 in the formula $N_4T_5 + N_5T_6 = T_4\widetilde{N_5} + T_5N_1$ which follows from (78), we obtain

$$\widetilde{N_5} = \begin{pmatrix} \sin^2\gamma & \sin\gamma\cos\gamma\cos\delta \\ -\sin\gamma\cos\gamma\cos\delta & -\cos^2\gamma \\ \sin\gamma\cos\gamma\sin\delta\cos(\phi-\psi) & \sin\gamma \\ cos\gamma\sin\delta\sin(\phi-\psi) & \\ -\sin\gamma\cos\gamma\sin\delta\sin(\phi-\psi) & \sin\gamma\cos\gamma\sin\delta\cos(\phi-\psi) \end{pmatrix},$$

hence $\phi = \psi$, hence $\gamma = \tilde{\gamma}$, $\delta = \tilde{\delta}$.

Thus, we have proved that

If an indecomposable H-normal operator acts in a space R^8 of rank 2 and has 2 eigenvalues: $\alpha \pm i\beta$ $(\alpha, \beta \in \Re$, $\beta > 0)$, then the pair $\{N, H\}$ is unitarily similar to the canonical pair $\{(54),(55)\}$.

We have considered all the possible alternatives for an indecomposable operator N and have obtained the canonical forms for each case. Thus, we have proved Theorem 2.

References

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